ELEN E3401: Electromagnetics

Spring 2025

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Lecture #2

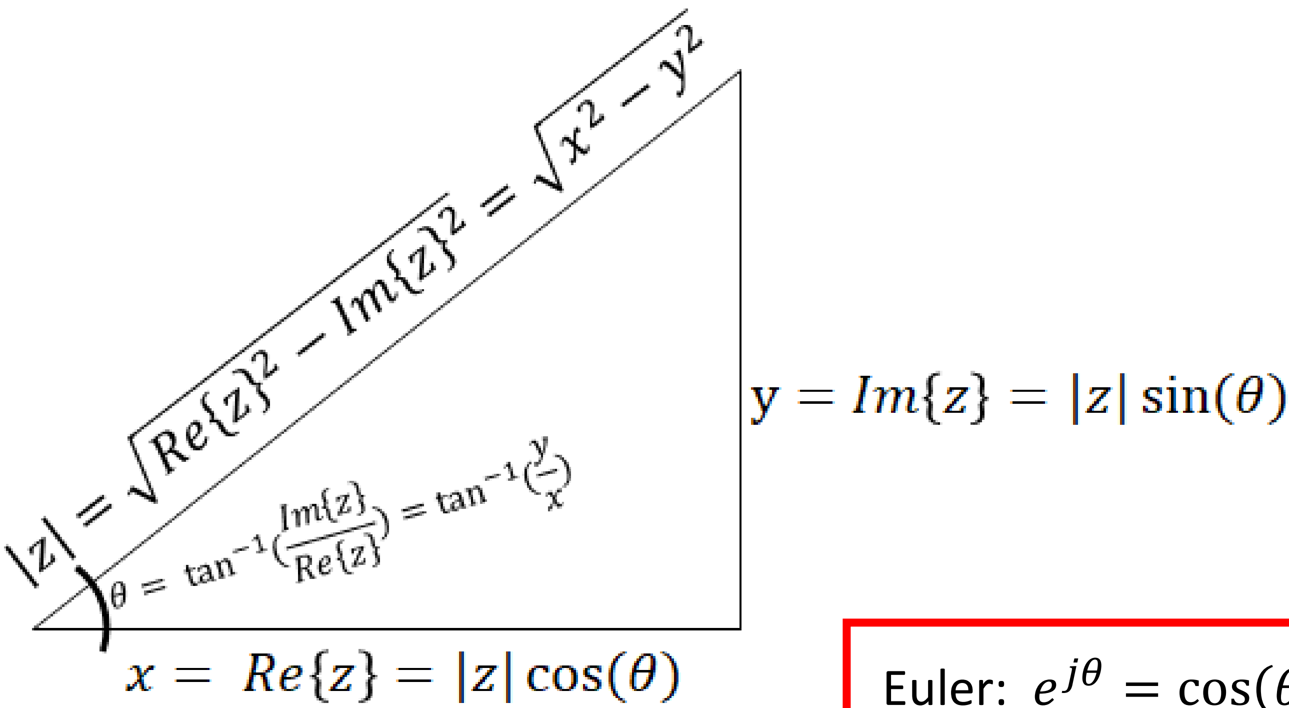
COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Complex Numbers and Algebra - Review

Complex number, z, is the sum of real and imaginary parts

Electrical Engineers use $j = \sqrt{-1}$

$$z = x + jy = |z|e^{j\theta} = |z|\angle\theta$$



Euler: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

Complex Numbers

- Complex Conjugate

$$z^* = (x + jy)^* = x - jy = |z|e^{-j\theta} = |z|\angle -\theta$$

$$|z| = \sqrt[+]{zz^*}$$

- Multiplication

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

OR

$$z_1 z_2 = |z_1|e^{j\theta_1}|z_2|e^{j\theta_2} = |z_1||z_2|e^{j(\theta_1+\theta_2)}$$

$$= |z_1||z_2|(\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$$

Complex Algebra

- Division $(z_2 \neq 0)$

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)}{(x_2 + jy_2)} \frac{x_2 - jy_2}{x_2 - jy_2} = \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} + \frac{j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

$$\mathrm{OR}$$

$$\frac{z_1}{z_2} = \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)}$$

- Exponentiate

$$z^n = \left(|z| e^{j\theta}\right)^n = |z|^n e^{jn\theta} = |z|^n (\cos(n\theta) + j \sin(n\theta))$$

$$z^{1/2} = \pm |z|^{1/2} e^{j\theta/2} = \pm |z|^{1/2} \left(\cos\left(\frac{\theta}{2}\right) + \mathrm{jsin}\left(\frac{\theta}{2}\right) \right)$$

Complex Relations

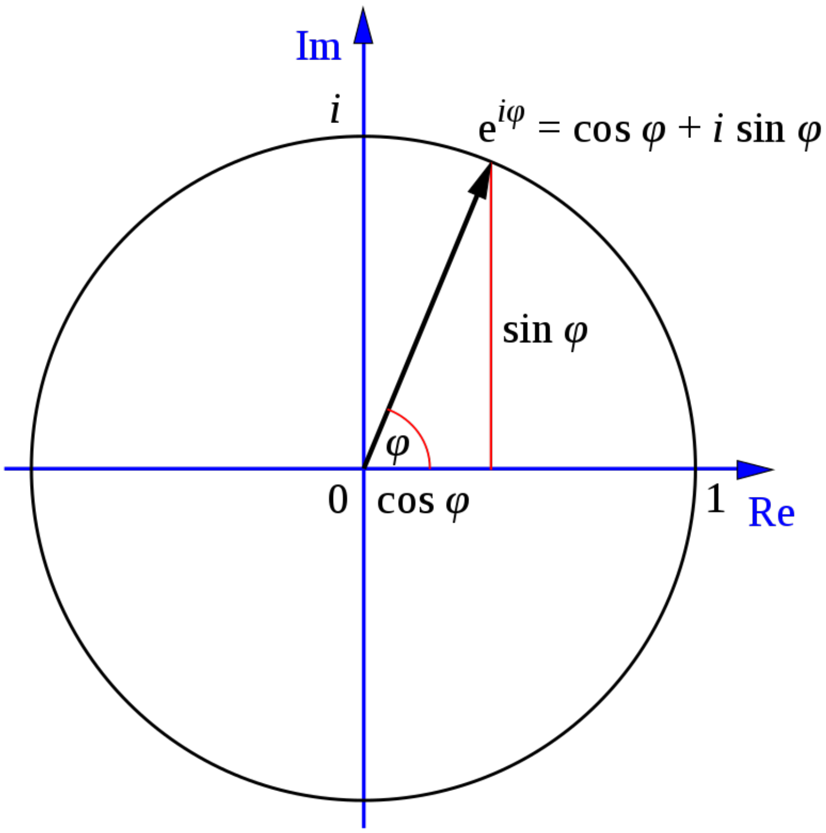
$$j = \sqrt{-1} = e^{j\pi/2} = 1\angle 90^o$$

$$-1 = j^2 = e^{\pm j} = 1\angle 180^o$$

$$-j = -e^{+j\pi/2} = +e^{-j\pi/2} = 1\angle -90^o$$

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = \pm e^{j\pi/4} = \pm\frac{(1+j)}{\sqrt{2}}$$

$$\sqrt{-j} = \left(e^{-j\pi/2}\right)^{1/2} = \pm e^{-j\pi/4} = \pm\frac{(1-j)}{\sqrt{2}}$$



What is $\sqrt{2j}$

Phasor Domain - Review

- Used to solve <u>linear systems</u> with periodic time function excitation
- Integral-differential time domain → linear equation in phasor domain
- Force function → expanded in Fourier series, superposition

$$v(t) = V_0 \cos(\omega t + \phi)$$

$$= \Re\left[V_0 e^{j(\omega t+\phi)}\right] = \Re\Big[\underbrace{V_0 e^{j\phi}}_{\text{Phasor } \tilde{V}} e^{j\omega t}\Big]$$

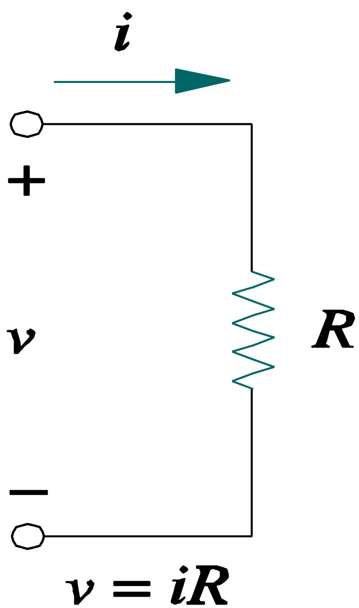
Time Domain $v(t)$ **Phasor Domain** $\tilde{V}$

$$v(t) = V_0 \cos(\omega t) \leftrightarrow \tilde{V} = V_0$$

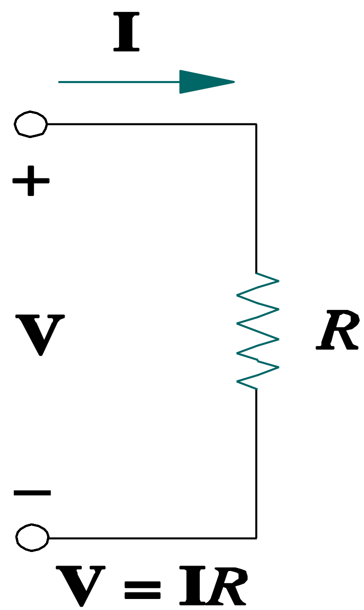
$$v(t) = V_0 \cos(\omega t + \phi) \leftrightarrow \tilde{V} = V_0 e^{j\phi}$$

Phasor Relation for Resistors

Time Domain



Phasor Domain



Current through resistor

Time Domain

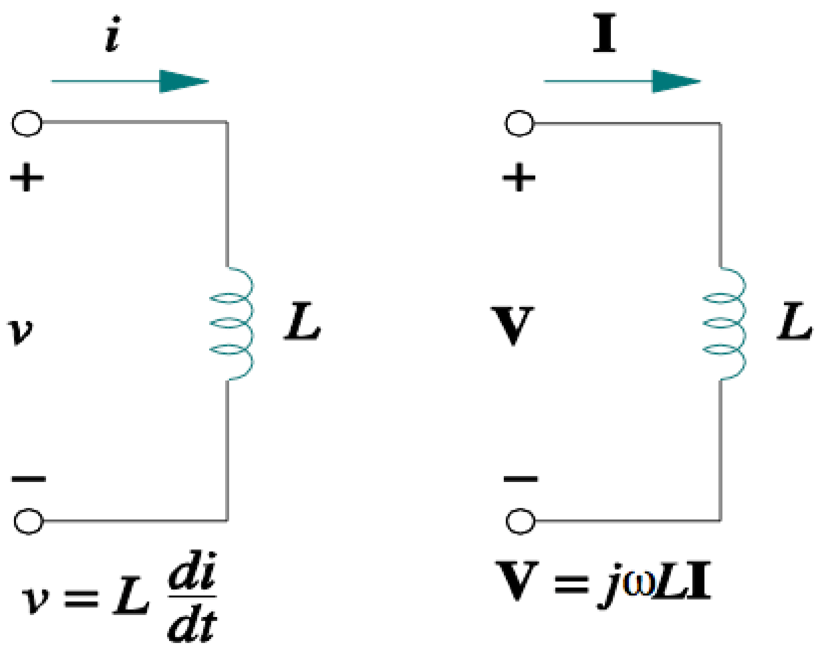
$$i = I_m \cos(\omega t + \phi)$$
$$v = iR = RI_m \cos(\omega t + \phi)$$

Phasor Domain

$$\tilde{V} = R\tilde{I} = RI_m\angle\phi$$

Phasor Relation for Inductors

Time Domain | Phasor Domain



Time Domain

$$v = L\frac{\partial i}{\partial t}$$

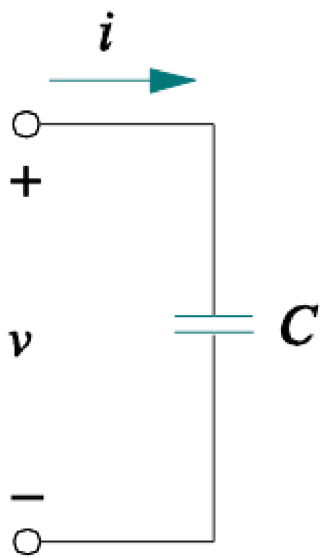
Phasor Domain

$$v_L = \Re\left[\tilde{V}_L e^{j\omega t}\right]$$
$$i_L = \Re\left[\tilde{I}_L e^{j\omega t}\right]$$
$$\Re\left[\tilde{V}_L e^{j\omega t}\right] = L\frac{\partial}{\partial t}\left[\Re\left(\tilde{I}_L e^{j\omega t}\right)\right]$$
$$= \Re\left[j\omega L\tilde{I}_L e^{j\omega t}\right]$$

$$\tilde{V}_L = j\omega L\tilde{I}_L$$
$$\tilde{Z}_L = \frac{\tilde{V}_L}{\tilde{I}_L} = j\omega L$$

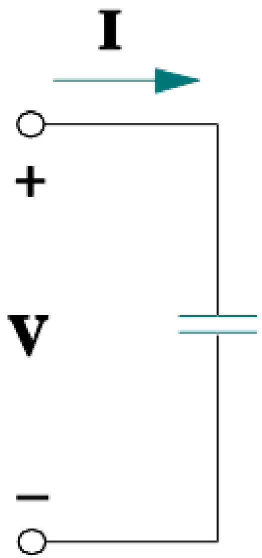
Phasor Relation for Capacitors

Time Domain



$$i = C\frac{dv}{dt}$$

Phasor Domain



$$\mathbf{I} = j\omega C\mathbf{V}$$

Time Domain

$$i = C\frac{\partial v}{\partial t}$$

Phasor Domain

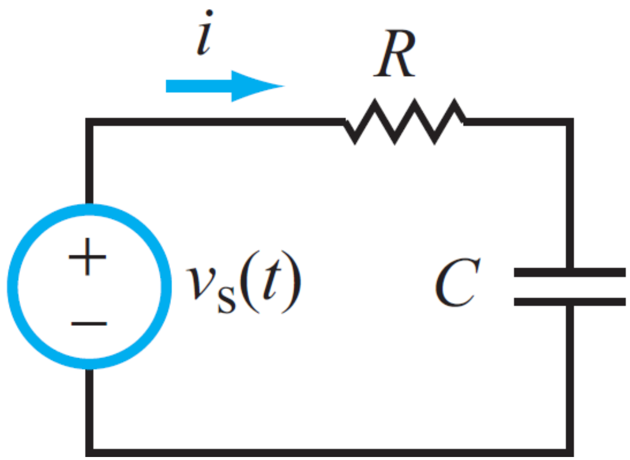
$$\tilde{I}_C = j\omega C\tilde{V}_C$$

$$\tilde{Z}_C = \frac{\tilde{V}_C}{\tilde{I}_C} = \frac{1}{j\omega C}$$

RC Circuit Phasor Solution

$v_s(t) = V_0 \sin(\omega t + \phi_0)$
V_0 = amplitude
ω = angular frequency
ϕ_0 = reference phase

Solve for: $i(t)$

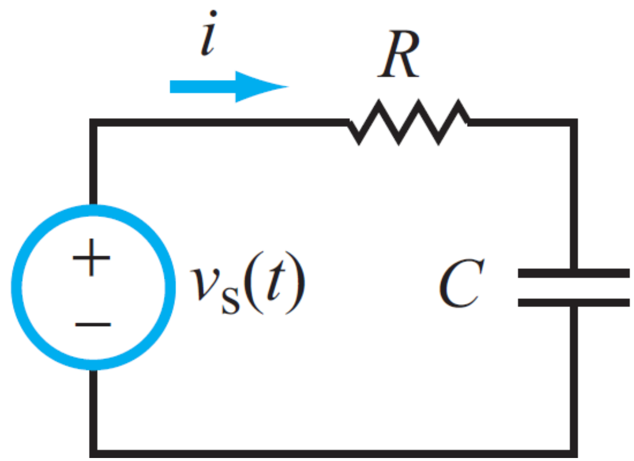


<u>Apply Kirchoff's Voltage Law (KVL):</u>

Time Domain: $Ri(t) + \frac{1}{C}\int i(t)\partial t = v_s(t)$

RC Circuit Phasor Solution

<u>Step 1: Move to Cosine reference:</u>



$\sin x = \cos(\pi/2 - x)$

$$v_s(t) = V_0 \sin(\omega t + \varphi_0)$$
$$= V_0 \cos\left(\frac{\pi}{2} - \omega t - \varphi_0\right)$$
$$v_s(t) = V_0 \cos\left(\omega t + \varphi_0 - \frac{\pi}{2}\right)$$

$\cos(x) = \cos(-x)$

$$v_s(t) = V_0 \sin(\omega t + \phi_0) = V_0 \cos\left(\frac{\pi}{2} - \omega t - \phi_0\right) = V_0 \cos\left(\omega t + \phi_0 - \frac{\pi}{2}\right)$$

RC Circuit Phasor Solution

<u>Step 2:</u> <u>Convert from time to phasor domain</u>

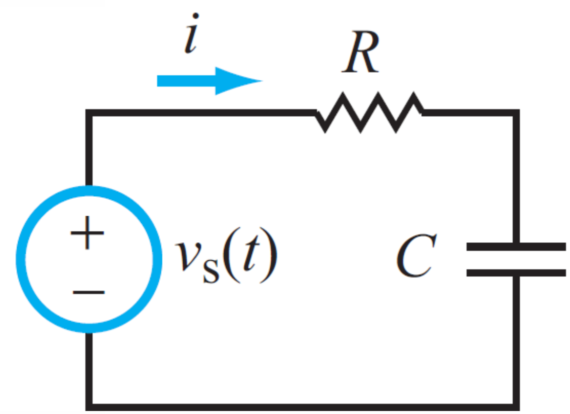
$$v_s(t) = V_0 \cos\left(\omega t + \phi_0 - \frac{\pi}{2}\right)$$

change time-dependent to phasors :

$$z(t) = \mathrm{Re}\left[\tilde{Z} e^{j\omega t}\right]$$

$\tilde{Z}$: time-independent function, phasor of instantaneous $z(t)$

$(\sim)$ → used to indicate instantaneous quantity



$$v_s(t) = \mathrm{Re}\left[V_0 e^{j(\omega t + \phi_0 - \pi/2)}\right]$$
$$= \mathrm{Re}\left[V_0 e^{j(\phi_0 - \pi/2)} e^{j\omega t}\right]$$
$$v_s(t) = \mathrm{Re}\left[\tilde{V}_s e^{j\omega t}\right]$$
$$\tilde{V}_s = V_0 e^{j(\phi_0 - \pi/2)}$$

phasor $\tilde{V}_s$ → corresponds to time function $v_s(t)$, has amplitude and phase → but independent of time

$$v_s(t) = V_0 \cos\left(\omega t + \phi_0 - \frac{\pi}{2}\right) = \Re\left[V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} e^{j\omega t}\right] \Rightarrow \tilde{V}_s = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)}$$

RC Circuit Phasor Solution

Time Domain: $Ri(t) + \frac{1}{C}\int i(t)\partial t = v_s(t)$

$$i(t) = \text{Re}\left[\tilde{I} e^{j\omega t}\right]$$

$$\frac{di}{dt} = \frac{d}{dt}\left[\text{Re}\left(\tilde{I} e^{j\omega t}\right)\right] = \text{Re}\left[\frac{d}{dt}\left(\tilde{I} e^{j\omega t}\right)\right]$$

$$\frac{di}{dt} = \text{Re}\left[j\omega \tilde{I} e^{j\omega t}\right]$$

$$\int i\,dt = \int \text{Re}\left(\tilde{I} e^{j\omega t}\right) dt = \text{Re}\left[\int \tilde{I} e^{j\omega t} dt\right]$$

$$\int i\,dt = \text{Re}\left[\frac{\tilde{I}}{j\omega} e^{j\omega t}\right]$$

$$\frac{di}{dt} \longleftrightarrow j\omega \tilde{I}$$

$$\int i\,dt \longleftrightarrow \frac{\tilde{I}}{j\omega}$$

RC Circuit Phasor Solution

Step 3: Obtain Phasor Form for Time-Domain Equation

$$R\, i(t) + \frac{1}{C}\int i(t)\, dt = v_s(t)$$

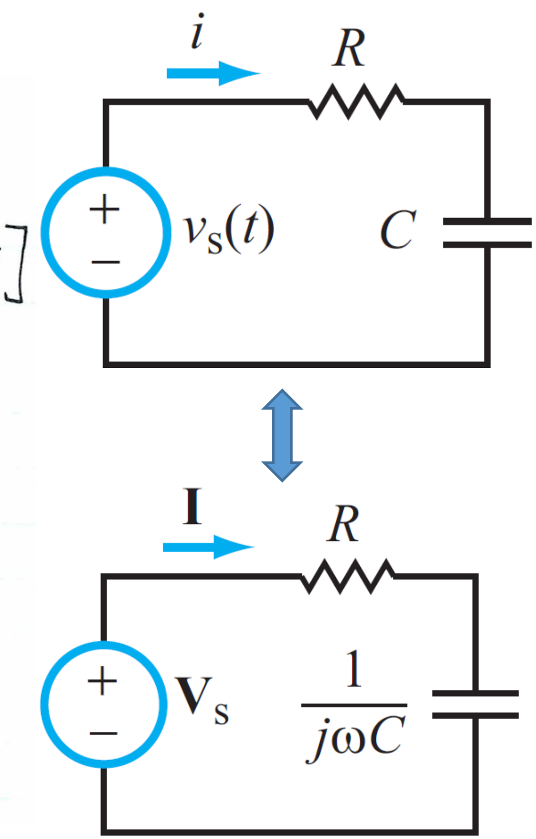
$$R \,\mathrm{Re}\left[\tilde{I} e^{j\omega t}\right] + \frac{1}{C}\mathrm{Re}\left[\frac{\tilde{I}}{j\omega} e^{j\omega t}\right] = \mathrm{Re}\left[\tilde{V}_s e^{j\omega t}\right]$$

$$\mathrm{Re}\left\{\left[\left(R + \frac{1}{j\omega C}\right)\tilde{I} - \tilde{V}_s\right] e^{j\omega t}\right\} = 0$$

since both Re and Im components $= 0$ and $e^{j\omega t} \neq 0$

$$\boxed{\tilde{I}\left(R + \frac{1}{j\omega C}\right) = \tilde{V}_s}$$

phasor domain



RC Circuit Phasor Solution

Step 4: Solve the Phasor Domain Equation

$$\tilde{I}\left(R+\frac{1}{j\omega C}\right)=\tilde{V}_s \quad\Rightarrow\quad \tilde{I}=\tilde{V}_s\Big/\left(R+\frac{1}{j\omega C}\right)=I_0 e^{j\phi}$$

$$\tilde{I}=\frac{\tilde{V}_s}{R+1/j\omega C} \Longrightarrow I_o e^{j\theta}$$

Real

$$\tilde{I}=V_o e^{j(\varphi_0-\pi/2)}\left[\frac{jwC}{1+j\omega RC}\right] \qquad j \rightarrow e^{j\pi/2}$$

$$|z|e^{j\theta} \leftrightarrow x+jy$$

$$1+j\omega RC \rightarrow \sqrt{1+\omega^2R^2C^2}\, e^{j\varphi_1}$$

$$\varphi_1=\tan^{-1}(\omega RC)$$

$$\tilde{I}=V_o e^{j(\varphi_0-\pi/2)}\left[\frac{e^{j\pi/2}(\omega C)}{\sqrt{1+\omega^2R^2C^2}\, e^{j\varphi_1}}\right]$$

$$\tilde{I}=\frac{V_o\,\omega C}{\sqrt{1+\omega^2R^2C^2}}\, e^{j(\varphi_0-\varphi_1)}$$

RC Circuit Phasor Solution

Step 5: Obtain the time domain solution *i*(t)

$$i(t) = \Re\left[\tilde{I}e^{j\omega t}\right]$$

$$= \Re\left[\frac{V_0\omega C}{\sqrt{1+\omega^2R^2C^2}}e^{j(\phi_0-\phi_1)}e^{j\omega t}\right]$$

$$= \frac{V_0\omega C}{\sqrt{1+\omega^2R^2C^2}}\cos(\omega t+\phi_0-\phi_1)$$

$$\boxed{i(t) = \frac{V_0\omega C}{\sqrt{1+\omega^2R^2C^2}}\cos(\omega t+\phi_0-\phi_1), \quad \phi_1 = \tan^{-1}(\omega RC)}$$

Time and Phasor Domain

$$v(t) = \Re\left[\tilde{V}e^{j\omega t}\right]$$

$v(t)$		$\tilde{V}$
$A\cos(\omega t)$	$\leftrightarrow$	A
$A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi\pm\pi)}$
$A\sin(\omega t)$	$\leftrightarrow$	$Ae^{-j\pi/2} = -jA$
$A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi-\pi/2)}$
$-A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi+\pi/2)}$
$A\cos(\omega t+\beta z+\phi_0)$	$\leftrightarrow$	$Ae^{j(\beta z+\phi_0)}$
$Ae^{-\alpha z}\cos(\omega t+\beta z+\phi_0)$	$\leftrightarrow$	$Ae^{-\alpha z}e^{j(\beta z+\phi_0)}$

Traveling waves in Phasor Domain

Much easier to deal with multiplying phasor domain exponentials than time domain integral-differential equation

Time and Phasor Domain

$$v(t) = \Re\left[\tilde{V} e^{j\omega t}\right]$$

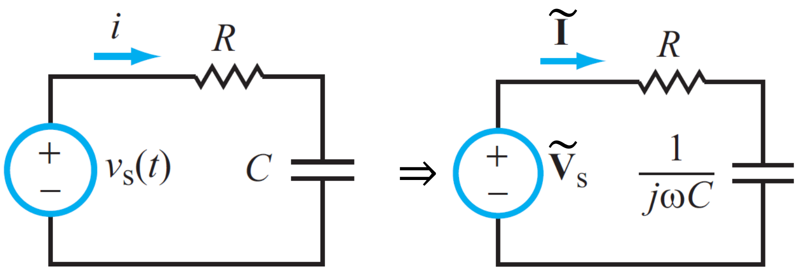
$v(t)$		$\tilde{V}$
$\frac{d}{dt}\left(v(t)\right)$	$\leftrightarrow$	$j\omega\tilde{V}$
$\frac{d}{dt}(A\cos(\omega t + \phi))$	$\leftrightarrow$	$j\omega A e^{j\phi}$
$\int v(t)dt$	$\leftrightarrow$	$(1/j\omega)\tilde{V}$
$\int A\cos(\omega t + \phi)\, dt$	$\leftrightarrow$	$(1/j\omega)Ae^{j\phi}$
$\int A\sin(\omega t + \phi)\, dt$	$\leftrightarrow$	$(1/j\omega)Ae^{j(\phi_0 - \pi/2)}$

ac Phasor Analysis: General Procedure

$$v_s(t) = V_0 \sin(\omega t + \phi_0)$$
$$= V_0 \cos\left(\omega t + \phi_0 - \frac{\pi}{2}\right)$$
$$= \Re\left[V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} e^{j\omega t}\right]$$
$$\Rightarrow \tilde{V}_s = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)}$$

1. Adopt Cosine Reference for Source

2. Transfer to Time-Independent Phasor Domain



$$i \rightarrow \tilde{I}$$
$$v \rightarrow \tilde{V}$$
$$R \rightarrow \tilde{Z}_R = R$$
$$L \rightarrow \tilde{Z}_L = j\omega L$$
$$C \rightarrow \tilde{Z}_C = 1/j\omega C$$

<u>Apply Kirchoff's Voltage Law (KVL)</u>

Time Domain: $Ri(t) + \frac{1}{C}\int i(t)\partial t = v_s(t)$

Phasor Domain: $\tilde{I}\left(\tilde{Z}_R + \tilde{Z}_C\right) = \tilde{I}\left(R + \frac{1}{j\omega C}\right) = \tilde{V}_s$

3. Obtain Phasor Form for Time Domain Equation

$$\tilde{I} = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} = I_0 e^{j\phi}$$

4. Solve Phasor Domain Equation for unknown Variable ($\tilde{I}$)

ac Phasor Analysis: General Procedure

4. Solve Phasor Domain Equation (cont')

Recall: $j = e^{j\pi/2}$

$$\tilde{I} = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \Rightarrow \tilde{I} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{1}{R + \frac{1}{j\omega C}} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{j\omega C}{1 + j\omega RC} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{e^{j\frac{\pi}{2}}\omega C}{1 + j\omega RC}$$

Recall: $x + jy \leftrightarrow |z|e^{j\phi}$

Denominator: $1 + j\omega RC = \sqrt{1 + \omega^2 R^2 C^2}\, e^{j\phi_1}, \qquad \phi_1 = \tan^{-1}(\omega RC)$

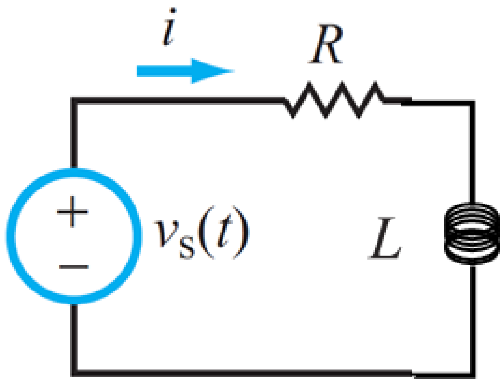
$$\tilde{I} = V_0 e^{j\left(\phi_0 - \frac{\pi}{2}\right)} \frac{e^{j\frac{\pi}{2}}\omega C}{\sqrt{1 + \omega^2 R^2 C^2}\, e^{j\phi_1}} = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j(\phi_0 - \phi_1)}$$

5. Transform Back to Time Domain

$$i(t) = \Re\left[\tilde{I} e^{j\omega t}\right] = \Re\left[\frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j(\phi_0 - \phi_1)} e^{j\omega t}\right] = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t + \phi_0 - \phi_1)$$

$$\boxed{i(t) = \frac{V_0 \omega C}{\sqrt{1 + \omega^2 R^2 C^2}} \cos(\omega t + \phi_0 - \phi_1), \qquad \phi_1 = \tan^{-1}(\omega RC)}$$

RL Circuit Phasor Example



Given $v_s(t)$
Find $i(t)$

Consider: $v_s(t) = 150\cos(\omega t)$

$$R = 400[\Omega]$$
$$L = 3[mH]$$
$$\omega = 10^5\left[\frac{rad}{s}\right]$$

<u>KVL</u>

Time Domain: $Ri(t) + L\frac{\partial i}{\partial t} = v_s(t)$

Phasor Domain: $R\tilde{I} + j\omega L\tilde{I} = \tilde{V}_s$

<u>Solve for</u> $\tilde{I} = \frac{\tilde{V}_s}{R + j\omega L}$ $\tilde{V}_s = 150\angle 0^o$

RL Circuit Phasor Example

$$\tilde{I} = \frac{150}{400 + j(10^5)(3 \times 10^{-3})} = \frac{150}{400 + j300}$$

$$\tan^{-1}(300/400) = 36.9^o = 0.6435\ [rad]$$

$$\tilde{I} = \frac{150}{\sqrt{400^2 + 300^2}e^{j36.9^o}} = \frac{150}{500e^{j36.9^o}}$$

$$\tilde{I} = \frac{150}{500}e^{-j36.9^o} = 0.3\angle - 36.9^o$$

$$i(t) = \Re\left[\tilde{I}e^{j\omega t}\right] = \Re\left[0.3e^{-j36.9^o}e^{j10^5 t}\right]$$

$$i(t) = 0.3\cos(10^5 t - 36.9^o)$$